Laminar Forced Convection in Eccentric Annuli

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The exact solution for the velocity distribution of fully developed laminar flow in an eccentric annulus is available in the literature. Snyder and Goldstein (1) used the general solution of the momentum equation in bipolar coordinates and obtained flow characteristics for a range of eccentricities and radius ratios. Heyda (2) determined the Green's function in bipolar coordinates for the potential equation of an eccentric annular region and used the result to obtain the solution of the momentum equation. The solution of the energy equation for forced convection problems in eccentric annular ducts of fully developed laminar flow with heat sources and constant wall temperature gradient, is difficult to obtain with the velocity distribution given in bipolar coordinates.

The purpose of this paper is to present a sufficiently accurate solution, without bipolar transformation, by a point matching method for fully developed laminar flow with uniform heat sources in eccentric annular ducts under the thermal boundary conditions of uniform heat flux from each wall so that at any axial position the wall temperatures are uniform. For doubly connected ducts, many combinations of thermal boundary conditions are possible, such as prescribed wall heat flux, wall temperature, and adiabatic wall. Since the method used can be adapted to various combinations of thermal boundary conditions at inner and outer walls, attention will be focused only to Dirichlet problems in this note. Admittedly, the thermal boundary condition selected for solution is highly restricted. The simple boundary condition was adopted to demonstrate the applicability of the method for solving a forced convective heat transfer problem involving eccentric annular region. The discussed laminar forced convection problems for several simply connected ducts were approached by Tao (3) using the method of complex variables. Recently, Sparrow and Haji-Sheikh (4) presented a point matching method by using the exact series solution in rectangular coordinates with the coefficients of the series solution determined by Gram-Schmidt orthonormalization for ducts of arbitrary shape with arbitrary thermal boundary conditions. This method is quite general and can be applied to arbitrary simply connected ducts. However, it appears to be difficult to apply to multiple connected ducts.

GOVERNING EQUATIONS

The governing equations and neglecting viscous dissipation effects (3) in polar coordinates for a steady, fully developed, constant property, laminar flow with heat sources are

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2}=\frac{a_1^2}{\mu}\frac{dp}{dz} \qquad (1)$$

$$\frac{\partial^2 \theta}{\partial r^2}+\frac{1}{r}\frac{\partial \theta}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \theta}{\partial \phi^2}=\left(\frac{a_1^2}{\alpha}\frac{\partial t}{\partial z}\right)u-\frac{a_1^2 Q}{k} \qquad (2)$$

The associated boundary conditions considered are

$$u=\theta=0 \quad \text{on } \Gamma \qquad (3)$$

ANALYSIS OF FULLY DEVELOPED LAMINAR FLOW WITH UNIFORM HEAT SOURCES IN ECCENTRIC ANNULI

The literature on laminar forced convection in annuli including entrance region is very extensive. However, to the author's knowledge, no results have been reported for

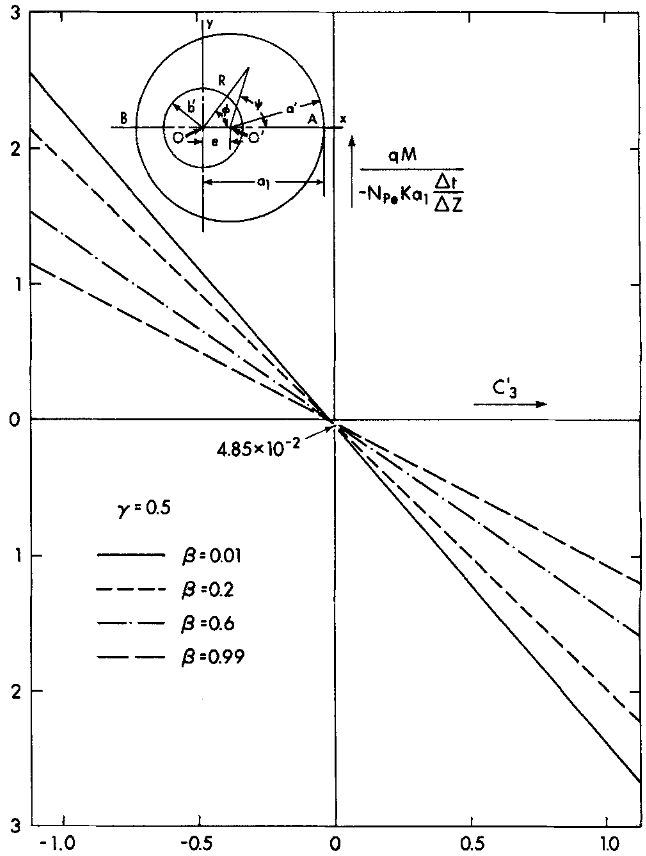

Fig. 1. Heat transfer rate vs. heat sources variable $c_3' = QMa_1/(-N_{Pe}k\Delta t/\Delta z)$ with eccentricity ratio β as a parameter for $\gamma = 0.5$ in. eccentric annuli.

the laminar heat transfer in an eccentric annulus. An analysis of slug flow heat transfer in an eccentric annulus was made by Snyder (5) when he used the solution obtained by El-Saden (6) for steady heat conduction with uniform heat sources in an eccentrically hollow cylinder. The exact solutions (7, 8) of Equations (1) and (2) can be written as

$$u=\left(-\frac{a_1^2}{\mu}\frac{dp}{dz}\right)\left[\frac{1}{4}\left(-r^2+b^2\frac{\ln r}{\ln b}\right)+C_0\left(1-\frac{\ln r}{\ln b}\right)+\sum_{j=1,2,3}^{\infty} C_j\left(r^j-\frac{b^{2j}}{r^j}\right)\cos j\phi\right] \qquad (4)$$

$$\theta=\left(-\frac{N_{Pe}\,a_1}{M}\frac{\Delta t}{\Delta z}\right)\Bigg\{\left[\frac{r^4}{64}-\frac{b^2r^2}{16\ln b}(\ln r-1)-\frac{C_0r^2}{4}\left(1-\frac{\ln r-1}{\ln b}\right)+\frac{3b^4}{64}-\frac{b^4}{16\ln b}+\frac{C_0b^2}{4\ln b}+E_0\ln\frac{r}{b}-C_1\left[\frac{r^3}{8}-\frac{b^2r}{2}\ln r-\frac{b^4}{2r}\left(\frac{1}{4}-\ln b\right)\right]\cos\phi-\sum_{j=2,3,4..}^{\infty}\frac{C_j}{4}\left[\frac{r^{j+2}}{j+1}+\frac{b^{2j}}{(j-1)r^{j-2}}-\frac{2j\,b^{2j+2}}{(j^2-1)r^j}\right]\cos j\phi$$

$$+\sum_{j=1,2,3..}^{\infty} E_j\left(r^j-\frac{b^{2j}}{r^j}\right)\cos j\phi\Bigg]-\left[\frac{Ma_1Q}{kN_{Pe}(\Delta t/\Delta z)}\right]$$

$$\left[\frac{1}{4}\left(-r^2+b^2\frac{\ln r}{\ln b}\right)+C_0\left(1-\frac{\ln r}{\ln b}\right)\right.$$

$$\left.+\sum_{j=1,2,3..}^{\infty} C_j\left(r^j-\frac{b^{2j}}{r^j}\right)\cos j\phi\right]\Bigg\} \quad (5)$$

where $N_{Pe}=\frac{u_m a_1}{\alpha}$, $u_m=-\frac{a_1^2}{\mu}\frac{dp}{dz}M$ and $\frac{\Delta t}{\Delta z}=\frac{\partial t}{\partial z}$

It is noted that the above solutions satisfy the boundary conditions at the inner wall. The unknown coefficients for the complementary solutions in Equations (4) and (5) were obtained by a point matching method that satisfies boundary conditions at twenty equally spaced points along the outer boundary AB (see Figure 1). The number 20 was obtained by trials after considering the cases with the largest eccentricity and extreme radius ratios and computing the boundary errors. For velocity and temperature the maximum boundary errors for the extreme cases are less than 10^{-5} of the maximum value inside the ducts. In most cases the maximum boundary errors are negligibly small, being 10^{-9} or less of the maximum value inside the duct. This observation confirms the convergence of the solution.

LAMINAR FLOW AND HEAT TRANSFER RESULTS

To ascertain the accuracy of the present method, various flow characteristics were computed, and excellent agreement was found when compared with the results given by Snyder and Goldstein (1), and Jonsson and Sparrow (9).

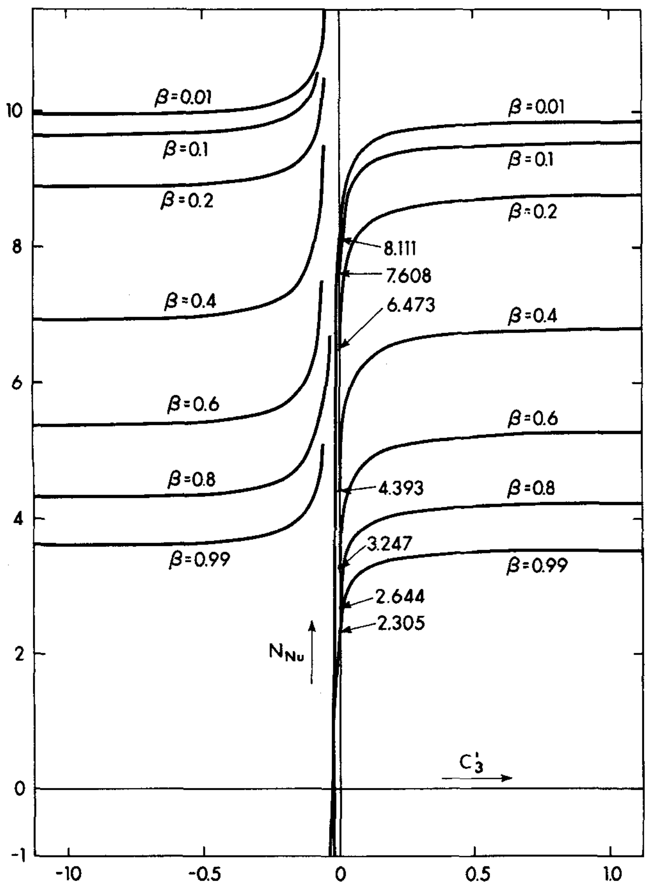

Fig. 2. Nusselt number vs. heat sources variable $c_3'=QMa_1/(-N_{Pe}k\Delta t/\Delta z)$ with eccentricity ratio β as a parameter for $\gamma=0.5$ in. eccentric annuli.

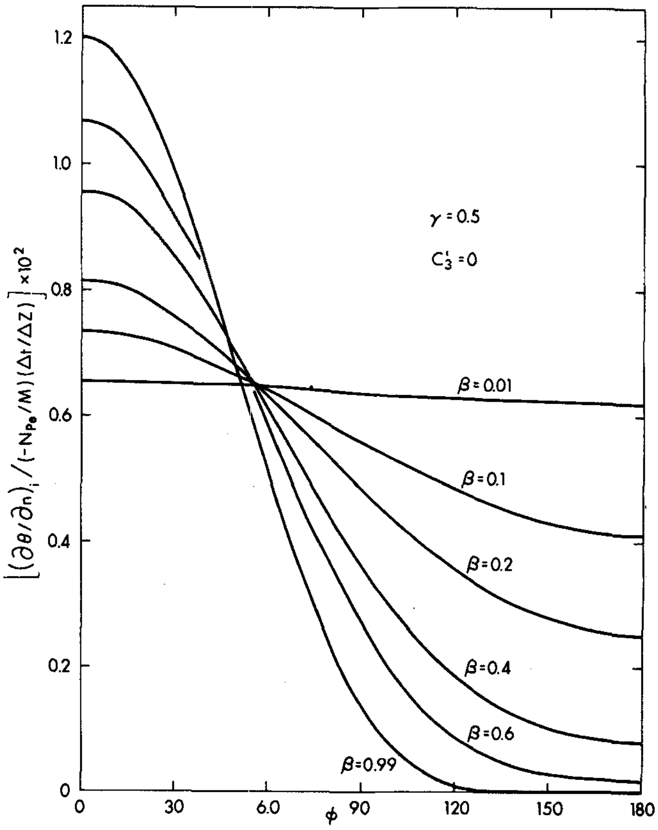

Fig. 3. Normal temperature gradient along inner wall for $\gamma=0.5$, $c_3'=0$ in eccentric annuli.

The heat transfer characteristics defined by the following equations (3) are usually of interest:

$$q=\left[\frac{1}{\alpha}\frac{\Delta t}{\Delta z}u_m-\frac{Q}{k}\right]kA \quad (6)$$

$$h=-q/S\theta_b \quad (7)$$

$$N_{Nu}=hD_e/k=-4(A/S)^2\left[\frac{1}{\alpha}\frac{\Delta t}{\Delta z}u_m-\frac{Q}{k}\right]\Big/\theta_b \quad (8)$$

The results are

$$q=\left(-N_{Pe}\frac{\Delta t}{\Delta z}\frac{ka_1}{M}\right)\pi(a^2-b^2)\left[M-\left(\frac{QMa_1}{-N_{Pe}(\Delta t/\Delta z)k}\right)\right] \quad (9)$$

$$h=-\frac{k}{a_1}\frac{a-b}{2}\left[M-\left(\frac{QMa_1}{-N_{Pe}(\Delta t/\Delta z)k}\right)\right]\Big/\theta_M \quad (10)$$

$$N_{Nu}=(a-b)^2\left[-M+\left(\frac{QMa_1}{-N_{Pe}(\Delta t/\Delta z)k}\right)\right]\Big/\theta_M \quad (11)$$

The dimensionless overall heat transfer rate at the wall and the Nusselt number defined above are plotted in Figures 1 and 2 against dimensionless heat sources parameter, c_3', for a range of eccentricities, β. For clarity, the Nusselt number for small negative values of c_3' is not shown in the figures. Figures 3 and 4 show the dimensionless normal temperature gradients at the inner and outer walls for the case $\gamma=0.5$ for a range of eccentricities, $\beta=0.01\sim0.99$.

It is noted that all the flow and heat transfer results were obtained by applying Simpson's rule for integration, as the exact integrations involving eccentric annular re-

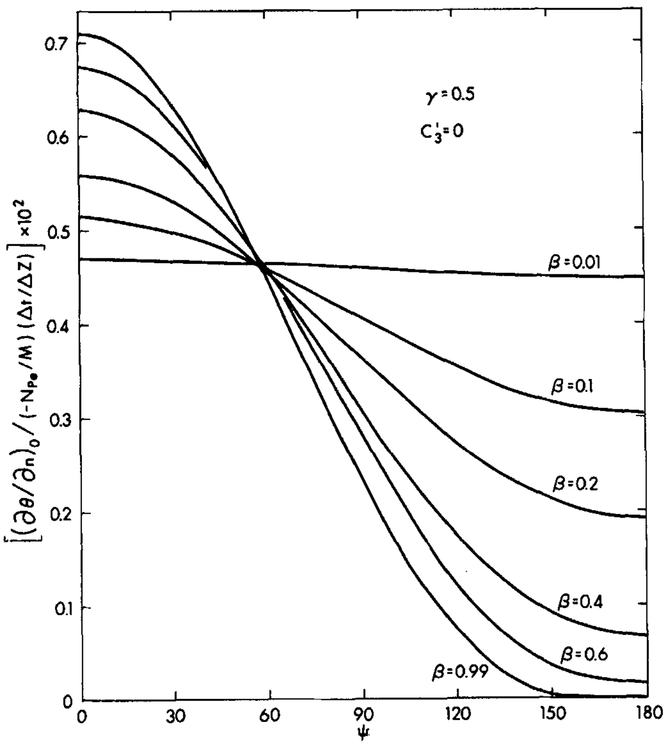

Fig. 4. Normal temperature gradient along outer wall for $\gamma = 0.5$, $c_3' = 0$ in eccentric annuli.

gion are impossible. For a two dimensional extension of Simpson's rule, twenty divisions were employed; this insures the accuracy to six significant figures. Some Nusselt number results with $c_3' = 0$ are listed in Table 1.

TABLE 1. SOME NUSSELT NUMBER RESULTS FOR ECCENTRIC ANNULI WITH $c_3' = 0$

Radius ratio γ	Eccentricity ratio β 0	0.01	0.10	0.20	0.40	0.60	0.80	0.90	0.99
0.25	—	7.800	7.419	6.524	4.761	3.735	3.203	3.038	2.925
0.50	8.117	8.111	7.608	6.473	4.393	3.247	2.644	2.446	2.305
0.75	—	8.208	7.659	6.432	4.227	3.024	2.384	2.171	2.016
0.90	8.232	8.226	7.667	6.422	4.192	2.975	2.324	2.106	1.947

CONCLUDING REMARKS

A study of the convergence of the solutions by the point matching method for the eccentric annular ducts shows that all the numerical results are very close to the exact values. This method yields sufficiently accurate results. Consequently, the least squares method of satisfying boundary conditions was not employed.

It may be of interest to know that the laminar forced convection in isosceles triangular ducts (*10*) can be approached by the point matching method.

ACKNOWLEDGMENT

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NOTATION

A = cross sectional area
a = dimensionless outer wall radius, a'/a_1
a' = outer wall radius
a_1 = $(e + a')$
b = dimensionless inner wall radius, b'/a_1
b' = inner wall radius
C_j = series coefficients for velocity
c_3' = dimensionless heat sources variable, $QMa_1/(-N_{Pe}k\Delta t/\Delta z)$
D_e = equivalent hydraulic diameter, $4A/S$
E_j = series coefficients for temperature
e = eccentricity (distance between centers)
h = fully developed heat transfer coefficient
j = summation index
k = thermal conductivity
M = constant, $u_m/(-a_1^2/\mu)(dp/dz)$
N_{Nu} = Nusselt number, hD_e/k
N_{Pe} = Peclet number, $u_m a_1/\alpha$
n = outward normal to the boundary
p = static pressure
Q = heat source intensity
q = overall heat transfer rate at wall per unit axial length
R = radial coordinates
r = dimensionless radial coordinate, R/a_1
S = circumference of cross section, $2\pi(a' + b')$
t = local temperature
u = local axial velocity
u_m = mean velocity
z = axial coordinate

Greek Letters

α = thermal diffusivity, $k/\rho c_p$
β = eccentricity ratio, $e/(a' - b')$
Γ = boundary curves of cross section
γ = radius ratio, b'/a'
Δ = finite difference
θ = temperature excess, $t - t_w$
θ_b = mixed mean temperature
θ_M = dimensionless mixed mean temperature, $\theta_b/(-N_{Pe}\,a_1/M(\Delta t/\Delta z))$
μ = viscosity
ρ = density
ϕ = angular coordinate (see Figure 1)
ψ = angular coordinate (see Figure 1)

Subscripts

i = inner wall
m = mean value
o = outer wall
w = wall

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